Amalgamated Rationality of Fuzzy Preference Relations Subject to Several Rationality Criteria.

Vincenzo CUTELLO
Department of Mathematics
University of Catania
Catania, Italy

Javier MONTERO
Faculty of Mathematics
Complutense University
Madrid, Spain

Abstract

Rationality or *consistency* of fuzzy preferences may simultaneously be estimated according to d ifferent criteria, although for practical purposes a single amalgamated value is usually obtained and analyzed. An axiomatic approach to *rationality*, viewed as a fuzzy property of fuzzy preferences, has been previously developed by the authors. It is now natural to ask if and when several rationality measures can be aggregated into a global rationality measure.

Keywords: aggregation rules, fuzzy preferences, decision making.

1 Introduction

Given a finite set of alternatives X, let us consider all complete fuzzy binary relations defined on X, that is, mappings $\mu : X \times X \rightarrow [0,1]$ such that each value $\mu(x,y)$ represents the intensity value to which alternative y is not worse than alternative x. We assume that μ is complete, i.e. $\mu(x,y) + \mu(y,x) \geq 1, \forall x, y \in X$ (incomparability problems would apprear otherwise).

An axiomatic characterization of the notion of *rationality* measure has been given in [2], where *consistency* was viewed as a fuzzy property of fuzzy binary preference relations. Since each fuzzy preference relation can be measured according to different rationality criteria, we should be willing to check whether or not we can combine different rationality criteria into one, at least by means of the two most frequent operators: *min* and *max*.

2 Fuzzy rationality measures

Max-min transitivity is a common rationality measure in the context of fuzzy binary preference relations. But is it obviously not the unique consistency criterion we can consider. Moreover, it is quite surprising that the property of being max-min transitive is crisp, not allowing any gradation in consistency intensity (each relation either is or is not max-min transitive, despite we intuitively realize that some fuzzy preferences are much closer to max-min transitivity than others).

In [2] rationality measures are formally characterized as mappings

$$\rho : \mathcal{P} \to [0,1]$$

where

$$\mathcal{P} = \bigcup_{X \text{ finite}} \mathcal{P}(X)$$

represents the family of all complete fuzzy binary relations. The following conditions were assumed in order for such mappings to define a *rationality measure*.

(R1) $\rho(\mu) = 1$ for any μ defining a crisp strict chain on X (there are no two distinct indifferent alternatives in a chain, and transitivity holds for crisp strict preference relation).

(R2) Given $\mu \in \mathcal{P}$ and a permutation $\pi : X \to X$ then

$$\rho(\mu^\pi) = \rho(\mu)$$

where $\mu^\pi(x,y) = \mu(\pi(x), \pi(y))$ for all $x, y \in X$.

(R3) For all $\mu \in \mathcal{P}$, $\rho(\neg\mu) = \rho(\mu)$, where $\neg\mu(x,y) = \mu(y,x)$ for all $x, y \in X$.

(R4) Let Y be a non-empty finite set of alternatives and let x be an extra alternative not belonging to Y. Let us consider a fuzzy preference $\mu : Y \times Y \to [0,1]$ such that $\mu(y,z) = 1, \mu(z,y) = 0, \forall y \in Y_1, \forall z \in Y_2$ for some Y_1, Y_2 partition of Y, and an extension μ' such that

$$\mu'(y,z) = \mu(y,z), \forall y, z \in Y$$
$$\mu'(y,x) = 1, \mu'(x,y) = 0, \forall y \in Y_1$$
$$\mu'(x,z) = 1, \mu'(z,x) = 0, \forall z \in Y_2$$
$$\mu'(x,x) = 1$$

Then it must be

$$\rho(\mu') \geq \rho(\mu).$$

(R5) Let $\mu \in \mathcal{P}(X)$ be fixed. Given an arbitrary ordered pair of alternatives $(\bar{x}, \bar{y})$, an arbitrary point $(\bar{a}, \bar{b}) \in [0,1] \times [0,1]$, and real numbers γ and λ such that $0 \leq \bar{a} + \lambda \cos\gamma \leq 1$, $0 \leq \bar{b} + \lambda \sin\gamma \leq 1$ and $a + \bar{b} + \lambda(\sin\gamma + \cos\gamma) \geq 1$, we denote by $\Gamma_\mu((\bar{x}, \bar{y}), (\bar{a}, \bar{b}), \gamma, \lambda)$ the fuzzy preference relation defined as

$$\Gamma_\mu((x,\bar{y}), (\bar{a}, \bar{b}), \gamma, \lambda)(x,y) = \begin{cases} \bar{a} + \lambda\cos\gamma & \text{if } (x,y) = (\bar{x}, y) \\ \bar{b} + \lambda \sin\gamma & \text{if } (x,y) = (\bar{y}, \bar{x}) \\ \mu(x,y) & \text{otherwise} \end{cases}.$$

Let $(\bar{x}, \bar{y})$, $(a, \bar{b})$, γ be fixed and let us consider the fuzzy preference relation $\Gamma^*(\lambda)$ defined as

$$\Gamma^*(\lambda)(x,y) = \Gamma_\mu((\bar{x}, \bar{y}), (\bar{a}, \bar{b}), \gamma, \lambda)(x,y)$$

Then, one of the following two properties must be verified by ρ.

(R5.1) there is no value λ such that

$$\begin{aligned} \rho(\Gamma^*(\lambda_1)) &> \rho(\Gamma^*(\lambda)) \\ \rho(\Gamma^*(\lambda)) &< \rho(\Gamma^*(\lambda_2)) \end{aligned}$$

for some λ_1, λ_2 such that $\lambda_1 < \lambda < \lambda_2$.

(R5.2) there is no value λ such that

$$\begin{aligned} \rho(\Gamma^*(\lambda_1)) &< \rho(\Gamma^*(\lambda)) \\ \rho(\Gamma^*(\lambda)) &> \rho(\Gamma^*(\lambda_2)) \end{aligned}$$

for some λ_1, λ_2 such that $\lambda_1 < \lambda < \lambda_2$.

We then have the following definition.

DEFINITION 2.1 *Any mapping* $\rho : \mathcal{P} \rightarrow [0,1]$ *is a*

1. pessimistic fuzzy rationality measure *if it verifies conditions (R1)-(R5.1);*
2. optimistic fuzzy rationality measure *if it verifies conditions (R1)-(R5.2).*

□

A fuzzy rationality measure being both pessimistic and optimistic will be said *normal*.

For example, classical max-min transitivity rationality condition appears as a *pessimistic* fuzzy rationality measure when written as

(Ex1)

$$\rho_{maxmin}(\mu) = \begin{cases} 1 & \text{if } \mu(x,z) \geq \min(\mu(x,y), \mu(y,z)), \forall x,y,z \in X. \\ 0 & \text{otherwise} \end{cases}$$

Some more examples of fuzzy rationality measures can be found in [3].

3 Equivalent rationality measures

As pointed out in [1], two fuzzy rationality measures can be considered equivalent if they always agree when comparing individual rationalities, from a qualitative point of view. In this case, it can sometimes be considered that discrepance between both rationality measures is just due to different underlying scales.

DEFINITION 3.1 *Given two fuzzy rationality measures* ρ_1 *and* ρ_2 *we will say that* ρ_1 *and* ρ_2 *are equivalent if and only if for any pair of individuals* μ_1 *and* μ_2, $\rho_1(\mu_1) \geq \rho_1(\mu_2)$ *if and only if* $\rho_2(\mu_1) \geq \rho_2(\mu_2)$. □

In this particular context, it was proven that any class of equivalent fuzzy rationality measures is closed with respect to the most important logical compositions.

THEOREM 3.1 *Let us consider* k *equivalent fuzzy rationality measures* $\rho_1, \rho_2, \cdots, \rho_k$. *Let* $H : [0,1]^k \rightarrow [0,1]$ *be a strictly nondecreasing mapping, in such a way that* $H(a_1, \cdots, a_k) \geq H(b_1, \cdots, b_k)$ *if* $a_j \geq b_j$

for all $j = 1, 2, \cdots, k$ and $H(a_1, \cdots, a_k) > H(b_1, \cdots, b_k)$ whenever $a_j > b_j$ for all $j = 1, 2, \cdots, k$. Let also assume $H(1, 1, \cdots, 1) = 1$. Then the mapping

$$H(\rho_1, \rho_2, \cdots, \rho_k) : \mathcal{P} \rightarrow [0, 1]$$

defined as

$$H(\rho_1, \rho_2, \cdots, \rho_k)(\mu) = H(\rho_1(\mu), \rho_2(\mu), \cdots, \rho_k(\mu)), \forall \mu$$

is a fuzzy rationality measure equivalent to $\rho_1, \rho_2, \cdots, \rho_k$.

Since OWA operators [5] do verify the conditions of the above theorem, they can be used to obtain new measures from finite collections of equivalent fuzzy rationality measures.

4 General results on *min* and *max* operators

The following results show the good performance of *min* and *max* operators when restricted to pessimistic and optimistic rationality measures.

THEOREM 4.1 *Let ρ_1 and ρ_2 be two pessimistic (optimistic) fuzzy rationality measures. Then the mapping $min(\rho_1, \rho_2)$ ($min(\rho_1, \rho_2)$) defines a pessimistic (optimistic) fuzzy rationality measure.*

An analogous closure result is obtained for OWA operators if restricted to more particular families of pessimistic or optimitic rationality measures.

THEOREM 4.2 *Let $\rho_1, \rho_2, \cdots, \rho_k$ be k fuzzy rationality measures, all of them pessimistic (optimistic), in such a way that -according to notation given in (R5)- every mapping $\rho_i(\Gamma^*(\lambda))$ is concave (convex) in λ. Then $H(\rho_1, \rho_2, \cdots, \rho_k)$ is also a pessimistic (optimistic) fuzzy rationality measure, if H is an OWA operator with increasing (decreasing) associated weights.*

5 Final Remarks

We point out the importance of the completeness assumption. Rationality under incomparability (as modeled in [4]) appears as an interesting and promising topic for future research.

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